Solution to Exercise 5

1. Find the partial derivatives of the following functions:

   (a) $(xy - 5z)/(1 + x^2)$,

   (b) $x/\sqrt{x^2 + y^2}$,

   (c) $\arctan y/x$,

   (d) $\log((t + 1)^3 + ts^2)$,

   (e) $\sin(xy^2z^3)$,

   (f) $|x|^\alpha, \quad x = (x_1, \cdots, x_n)$.

   Solution. (a) $\dfrac{\partial}{\partial x} = \dfrac{y - x^2y + 10xz}{(1 + x^2)^2}$, $\quad \dfrac{\partial}{\partial y} = \dfrac{x}{1 + x^2}$, $\quad \dfrac{\partial}{\partial z} = \dfrac{-5}{1 + x^2}$.

   (b) $\dfrac{\partial}{\partial x} = \dfrac{y^2}{(x^2 + y^2)^{3/2}}$, $\quad \dfrac{\partial}{\partial y} = \dfrac{-xy}{(x^2 + y^2)^{3/2}}$.

   (c) $\dfrac{\partial}{\partial x} = \dfrac{-y}{x^2 + y^2}$, $\quad \dfrac{\partial}{\partial y} = \dfrac{x}{x^2 + y^2}$.

   (d) $\dfrac{\partial}{\partial t} = \dfrac{3(t + 1)^2 + s^2}{(t + 1)^3 + ts^2}$, $\quad \dfrac{\partial}{\partial s} = \dfrac{2ts}{(t + 1)^3 + ts^2}$.

   (e) $\dfrac{\partial}{\partial x} = y^2z^3 \cos xy^2z^3$, $\quad \dfrac{\partial}{\partial y} = 2xyz^3 \cos xy^2z^3$, $\quad \dfrac{\partial}{\partial z} = 3xy^2z^2 \cos xy^2z^3$.

   (f) $\dfrac{\partial}{\partial x_j} = \alpha x_j|x|^{\alpha-2}, \quad j = 1, \cdots, n.$

2. Verify $f_{xy} = f_{yx}$ for the following functions:

   (a) $x \cos y + e^{2y}$,

   (b) $x \log(1 + y^2) - \sin(xy)$,

   (c) $(x + y)/(x^5 - y^9)$.

   Solution. Omitted.

3. Consider the function

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}, \quad (x, y) \neq (0, 0) ,$$

   and $f(0, 0) = 0$. Show that f_{xy} and f_{yx} exist but are not equal at $(0, 0)$.

   Solution. For $(x, y) \neq (0, 0)$,

$$f_x = \frac{x^4y + 4x^2y^3 - y^5}{(x^2 + y^2)^2} ,$$

$$f_y = \frac{x^5 - 4x^3y^2 - xy^4}{(x^2 + y^2)^2} .$$

When $(x, y) = (0, 0)$, $f_x(0, 0) = 0$ and $f_y(0, 0) = 0$. We have

$$f_{xy}(0,0) = \lim_{y\to 0} \frac{f_x(0,y) - f_x(0,0)}{y - 0} = -1,$$

but

$$f_{yx}(0,0) = \lim_{x\to 0} \frac{f_y(x,0) - f_y(0,0)}{x - 0} = 1.$$

They are not equal at $(0, 0)$.

4. Find

$$\frac{\partial^3 u}{\partial x \partial y \partial z}\ , \qquad \text{where } u(x, y, z) = e^{xyz}\ .$$

Solution. $(1 + 3xyz + x^2y^2z^2)e^{xyz}$.

5. * Show that

$$\frac{\partial^{m+n} v}{\partial x^m \partial y^n} = \frac{2(-1)^m(m + n - 1)!(mx + ny)}{(x - y)^{m+n+1}}\ ,$$

where

$$v(x, y) = \frac{x + y}{x - y}\ .$$

Solution. First show it is true for all m and $n = 0$ and then use induction on n.

6. *

(a) A harmonic function is a function satisfies the Laplace equation

$$\Delta u \equiv \left(\frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_n}\right) u = 0\ .$$

Show that all n-dimensional harmonic functions form a vector space.

(b) Find all harmonic functions which are polynomials of degree ≤ 2 for the two dimensional Laplace equations. Show that they form a subspace and determine its dimension.

Solution. (a) Let u and v be two harmonic functions. By linearity, we have

$$\Delta(\alpha u + \beta v) = \alpha \Delta u + \beta \Delta v = 0\ ,$$

so all harmonic functions form a vector space.

(b) Let $p(x, y) = a + bx + cy + dx^2 + 2exy + fy^2$ be a general polynomial of degree 2. If it is harmonic,

$$0 = \Delta p(x, y) = \Big(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\Big) p(x, y) = 2d + 2f = 0\ .$$

Therefore, it is harmonic if and only $d = -f$. Writing in the form

$$p(x, y) = a + bx + cy + d(x^2 - y^2) + 2exy\ ,$$

we see that the space of all harmonic polynomials of degree ≤ 2 is spanned by $1, x, y, x^2 - y^2$, and xy. These five functions are linearly independent, so the dimension of this subspace is 5.

7. Consider the function
$$g(x,y) = \sqrt{|xy|}\ .$$
Show that g_x and g_y exist but g is not differentiable at $(0,0)$.

Solution. $g_x(0,0) = \lim_{x\to 0} \dfrac{g(x,0)-g(0,0)}{x} = 0.$
Similarly, $g_y(0,0) = 0$. Therefore, the differential of g at $(0,0)$ vanishes identically. We have
$$\frac{g(x,y)-0}{\sqrt{x^2+y^2}} = \frac{\sqrt{|xy|}}{\sqrt{x^2+y^2}}\ ,$$
where is clearly not convergent to 0 as $(x,y)\to(0,0)$. Therefore, g is not differentiable at $(0,0)$.

8. Consider the function $h(x,y) = 1$ for (x,y) satisfying $x^2 < y < 4x^2$ and $h(x,y) = 0$ otherwise. Show that h_x and h_y exist but h is not differentiable at $(0,0)$.

Solution.
$$h_x(0,0) = \lim_{x\to 0}\frac{h(x,0)-h(0,0)}{x} = 0\ .$$
$$h_y(0,0) = \lim_{y\to 0}\frac{h(0,y)-h(0,0)}{y} = 0\ .$$
Therefore, h_x and h_y exist at $(0,0)$. However, h is not differentiable at $(0,0)$ since it is not even continuous at $(0,0)$.

9. Consider the function $j(x,y) = (x^2+y^2)\sin(x^2+y^2)^{-1}$ for $(x,y)\neq(0,0)$ and $j(0,0) = 0$. Show that it is differentiable at $(0,0)$ but its partial derivatives are not continuous there.

Solution.
$$j_x(0,0) = \lim_{x\to 0}\frac{j(x,0)-j(0,0)}{x} = \lim_{x\to 0} x\sin\frac{1}{x^2} = 0\ .$$
Similarly, $j_y(0,0) = 0$. If j is differentiable at $(0,0)$, its differential must vanish there. We have
$$\left|\frac{j(x,y)-0}{\sqrt{x^2+y^2}}\right| = \left|\sqrt{x^2+y^2}\sin\frac{1}{x^2+y^2}\right| \le \sqrt{x^2+y^2}\to 0\ ,$$
as $(x,y)\to(0,0)$, which shows that j is differentiable at $(0,0)$.

Next, for $(x,y)\neq(0,0)$,
$$j_x(x,y) = 2x\sin\frac{1}{x^2+y^2} - \frac{2x}{x^2+y^2}\cos\frac{1}{x^2+y^2}\ .$$
When $(x,0)\to(0,0)$,
$$j_x(x,0) = 2x\sin\frac{1}{x^2} - \frac{2}{x}\cos\frac{1}{x^2},$$
which does not tend to $j_x(0,0) = 0$. Therefore, j_x is not continuous at $(0,0)$. Similarly, j_y is also not continuous at $(0,0)$.

10. Use the Chain Rule to compute the first and second derivatives of the following functions.

(a) $f(x+y, x-y)$,

(b) $g(x/y, y/z)$,

(c) $h(t, t^2, t^3)$,

(d) $f(r\cos\theta, r\sin\theta)$,

Solution.

(a) $\tilde{f}(x,y) = f(x+y, x-y)$.
$\tilde{f}_x = f_x + f_y$,
$\tilde{f}_y = f_x - f_y$.
$\tilde{f}_{xx} = f_{xx} + f_{xy} + f_{yx} + f_{yy} = f_{xx} + 2f_{xy} + f_{yy}$,
$\tilde{f}_{xy} = f_{xx} - f_{xy} + f_{yx} - f_{yy} = f_{xx} - f_{yy}$,
$\tilde{f}_{yy} = f_{xx} - f_{xy} - f_{yx} + f_{yy} = f_{xx} - 2f_{xy} + f_{yy}$.

(b) $\tilde{g}(x,y) = g(u,v) = g(x/y, y/z)$.
$\tilde{g}_x = g_u \frac{1}{y} = g_u(x/y, y/z)\frac{1}{y}$,
$\tilde{g}_y = g_u \frac{-x}{y^2} + g_v \frac{1}{z} = g_u(x/y, y/z)\frac{-x}{y^2} + g_v(x/y, y/z)\frac{1}{z}$,
$\tilde{g}_z = g_v \frac{-y}{z^2} = g_v(x/y, y/z)\frac{-y}{z^2}$.
$\tilde{g}_{xx} = g_{uu}\frac{1}{y}\frac{1}{y} = \frac{1}{y^2} g_{uu}(x/y, y/z)$,
$\tilde{g}_{xy} = (g_{uu}\frac{-x}{y^2} + g_{uv}\frac{1}{z})\frac{1}{y}$,
$\tilde{g}_{yy} = g_u \frac{2x}{y^3} - (g_{uu}\frac{-x}{y^2} + g_{uv}\frac{1}{z})\frac{x}{y^2} + g_{vu}\frac{1}{z}(\frac{-x}{y^2}) + g_{vv}\frac{1}{z}\frac{1}{z} = g_u\frac{2x}{y^3} + g_{uu}\frac{x^2}{y^4} - g_{uv}\frac{2x}{y^2 z} + g_{vv}\frac{1}{z^2}$,
$\tilde{g}_{yz} = g_{uv}\frac{-x}{y^2}\frac{-y}{z^2} - g_v\frac{1}{z^2} + g_{vv}\frac{1}{z}\frac{-y}{z^2} = g_{uv}\frac{x}{yz^2} - g_v\frac{1}{z^2} - g_{vv}\frac{y}{z^3}$,
$\tilde{g}_{zz} = g_{vv}\frac{y^2}{z^4}$.

(c) $\tilde{h}(t) = h(x,y,z) = h(t, t^2, t^3)$.
$\tilde{h}'(t) = h_x + 2th_y + 3t^2 h_z$,
$\tilde{h}''(t) = (h_{xx} + 2th_{xy} + 3t^2 h_{xz}) + (2h_y + 2t(h_{yx} + 2th_{yy} + 3t^2 h_{yz}) + 6th_z + 3t^2(h_{zx} + 2th_{zy} + 3t^2 h_{zz})$
$= h_{xx} + 4th_{xy} + 6t^2 h_{xz} + 2h_y + 4t^2 h_{yy} + 12t^2 h_{yz} + 6th_z + 9t^4 h_{zz}$.

(d) $\tilde{f}(r,\theta) = f(r\cos\theta, r\sin\theta) = f(x,y)$.
$\tilde{f}_r = f_x\cos\theta + f_y\sin\theta$,
$\tilde{f}_\theta = -f_x r\sin\theta + f_y r\cos\theta$.
$\tilde{f}_{rr} = (f_{xx}\cos^2\theta + f_{xy}\cos\theta\sin\theta) + (f_{yx}\sin\theta\cos\theta + f_{yy}\sin^2\theta)$
$= f_{xx}\cos^2\theta + 2f_{xy}\cos\theta\sin\theta + f_{yy}\sin^2\theta$,
$\tilde{f}_{r\theta} = (-f_{xx}r\sin\theta + f_{xy}r\cos\theta)\cos\theta + f_x(-\sin\theta) + (-f_{yx}r\sin\theta + f_{yy}r\cos\theta)\sin\theta + f_y\cos\theta$
$= -rf_{xx}\sin\theta\cos\theta + f_{xy}r(\cos^2\theta - \sin^2\theta) + rf_{yy}\cos\theta\sin\theta - f_x\sin\theta + f_y\cos\theta$,
$\tilde{f}_{\theta\theta} = (f_{xx}r\sin\theta - f_{xy}r\cos\theta)r\sin\theta - f_x r\cos\theta + (-f_{yx}r\sin\theta + f_{yy}r\cos\theta)r\cos\theta - f_y r\sin\theta$
$= f_{xx}r^2\sin^2\theta - 2f_{xy}r^2\cos\theta\sin\theta - f_x r\cos\theta + f_{yy}r^2\cos^2\theta - f_y r\sin\theta$.

11. * Let $f(x,y)$ and $\varphi(x)$ be continuously differentiable functions and define

$$G(x) = \int_0^{\varphi(x)} f(x,y)\,dy \ .$$

Establish the formula

$$G'(x) = \int_0^{\varphi(x)} f_x(x,y)dy + f(x,\varphi(x))\varphi'(x) \ .$$

Hint: Consider the function

$$F(x,t) = \int_0^t f(x,y)dy \ .$$

Solution. Let $F(x,t) = \int_0^t f(x,y)dy$. Then $G(x) = F(x,\varphi(x))$. Therefore,

$$\begin{aligned} G'(x) &= F_x(x,\varphi(x)) + F_t(x,\varphi(x))\varphi'(x) \\ &= \int_0^{\varphi(x)} f_x(x,y)dy + f(x,\varphi(x))\varphi'(x) \end{aligned}$$

12. (a) Show that the ordinary differential equation satisfied by the solution of the Laplace equation in two dimension $\Delta u = 0$ when u depends only on the radius, that is,

$$u = f(r), \quad r = \sqrt{x^2+y^2} \ ,$$

is given by

$$f''(r) + \frac{1}{r}f'(r) = 0 \ .$$

(b) Can you find all these radially symmetric harmonic functions?

Solution.

(a) Write $u(x,y) = f(\sqrt{x^2+y^2})$. Then $\Delta u = 0$ is turned into

$$f''(r) + \frac{1}{r}f'(r) = 0 \ .$$

(b) This equation can be written as $(rf')' = 0$ which is readily integrated to $rf' = c_1$ for some constant c_1. i.e.

$$f' = \frac{c_1}{r}.$$

We conclude that all solutions are given by $f(r) = c_1 \log r + c_2$ for some constants c_1, c_2.

13. (a) Show that the ordinary differential equation satisfied by the solution of the Laplace equation in three dimension $\Delta u = 0$ when u depends only on the radius, that is,

$$u = f(r), \quad r = \sqrt{x^2+y^2+z^2} \ ,$$

is given by

$$f''(r) + \frac{2}{r}f'(r) = 0 \ .$$

(b) Can you find all these radially symmetric harmonic functions?

Solution. $u(x,y,z) = f(\sqrt{x^2+y^2+z^2})$. Then $\Delta u = 0$ is turned into

$$f''(r) + \frac{2}{r}f'(r) = 0 \ .$$

This equation can be written as $(r^2 f')' = 0$ which is readily integrated to $r^2 f' = -c_1$ for some constant c_1. We conclude that all solutions are given by $f(r) = \dfrac{c_1}{r} + c_2$ for some constants c_1, c_2.

14. Consider the one dimensional heat equation

$$u_t = u_{xx} \ .$$

(a) Show that $u(x,t) = v(y), y = x/\sqrt{t}$, solves this equation whenever v satisfies

$$v_{yy} + \frac{1}{2} y v_y = 0 \ .$$

(b) Show that $u(x,t) = e^{xt+2t^3/3} w(y), y = x+t^2$, solves this equation whenever w satisfies

$$w_{yy} = yw \ .$$

Solution. (a) We have

$$u_t = -\frac{1}{2}\frac{x}{t^{3/2}} v_y, \quad u_x = \frac{1}{\sqrt{t}} v_y, \quad u_{xx} = \frac{1}{t} v_{yy} \ ,$$

and the result follows.
(b) Let $E = e^{xt+2t^3/3}$. We have

$$u_t = ((x+2t^2)w + 2tw_y)E, \quad u_x = (tw + w_y)E, \quad u_{xx} = (t^2 w + 2tw_y + w_{yy})E \ ,$$

and the results follows.

15. * Let u be a solution to the two dimensional Laplace equation. Show that the function

$$v(x,y) = u\left(\frac{x}{x^2+y^2}, \frac{y}{x^2+y^2}\right)$$

also solves the same equation. Hint: Use $\Delta \log r = 0$ where $r = \sqrt{x^2+y^2}$.
Solution. Let $r = (x^2+y^2)^{1/2}$. We have

$$v_x = u_x (\log r)_{xx} + u_y (\log r)_{xy},$$

$$v_{xx} = \big(u_{xx}(\log r)_{xx} + u_{xy}(\log r)_{xy}\big)(\log r)_{xx} + u_x (\log r)_{xxx} +$$
$$\big(u_{yx}(\log r)_{xx} + u_{yy}(\log r)_{xy}\big)(\log r)_{xy} + u_y (\log r)_{xxy} \ ,$$
$$v_y = u_x (\log r)_{xy} + u_y (\log r)_{yy} \ ,$$
$$v_{yy} = \big(u_{xx}(\log r)_{xy} + u_{xy}(\log r)_{yy}\big)(\log r)_{xy} + u_x (\log r)_{xy} +$$
$$\big(u_{xy}(\log r)_{xx} + u_{yy}(\log r)_{yy}\big)(\log r)_{yy} + u_y (\log r)_{yyy} \ .$$

The key is $\Delta \log r = 0$. Using it we have

$$\Delta v(x,y) = \big((\log r)^2_{xx} + (\log r)^2_{xy}\big) \Delta u(x/(x^2+y^2)^{1/2}, y/(x^2+y^2)^{1/2}),$$

and the desired conclusion follows. This formula shows how to get a new harmonic function from an old one. It is called the Kelvin's transform.

16. * Express the differential equation

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0 \ ,$$

in the new variables

$$\xi = x, \quad \eta = x^2 + y^2.$$

Can you solve it?

Solution. Write $z(x, y) = \tilde{z}(\xi, \eta) = \tilde{z}(x, x^2 + y^2)$. Using $z_x = \tilde{z}_\xi + 2x\tilde{z}_\eta$ and $z_y = 2y\tilde{z}_\eta$ to get

$$y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = y(\tilde{z}_\xi + 2x\tilde{z}_\eta) - x(2y\tilde{z}_\eta) = y\tilde{z}_\xi \ .$$

The equation becomes $y\tilde{z}_\xi = 0$, i.e. $\tilde{z}$ depends on η only. The general solution is $f = f(x^2 + y^2)$, i.e. radially symmetric.

17. Express the one dimensional wave equation

$$\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} = 0 \ , \quad c > 0 \text{ a constant} \ ,$$

in the new variables

$$\xi = x - ct, \quad \eta = x + ct \ .$$

Then show that the general solution to this equation is

$$f(x, y) = \varphi(x - ct) + \psi(x + ct) \ ,$$

where φ and ψ are two arbitrary twice differentiable functions on $\mathbb{R}$.

Solution. Write $f(x, t) = \tilde{f}(\xi, \eta) = \tilde{f}(x-ct, x+ct)$. We have $f_x = \tilde{f}_\xi + \tilde{f}_\eta$, $f_t = -c\tilde{f}_\xi + c\tilde{f}_\eta$, $f_{xx} = \tilde{f}_{\xi\xi} + 2\tilde{f}_{\xi\eta} + \tilde{f}_{\eta\eta}$, and $f_{tt} = c^2\tilde{f}_{\xi\xi} - 2c^2\tilde{f}_{\xi\eta} + c^2\tilde{f}_{\eta\eta}$.
Therefore,

$$\frac{\partial^2 f}{\partial t^2} - c^2\frac{\partial^2 f}{\partial x^2} = (c^2\tilde{f}_{\xi\xi} - 2c^2\tilde{f}_{\xi\eta} + c^2\tilde{f}_{\eta\eta}) - c^2(\tilde{f}_{\xi\xi} + 2\tilde{f}_{\xi\eta} + \tilde{f}_{\eta\eta}) = -4c^2\tilde{f}_{\xi\eta}.$$

The differential equation is transformed to the new equation

$$\tilde{f}_{\xi\eta} = 0.$$

Now, $(\tilde{f}_\xi)_\eta = 0$ implies $\tilde{f}_\xi$ is independent of η. Therefore, $\tilde{f}_\xi = \varphi_1(\xi)$ for some φ_1 and hence $\tilde{f} = \int \varphi_1(\xi) + \varphi_2(\eta)$, i.e. $f(x, y) = \varphi(\xi) + \psi(\eta)$.

18. Consider the Black-Scholes equation

$$V_t + \frac{1}{2}\sigma^2 x^2 V_{xx} + rxV_x - rV = 0 \ .$$

(a) Show that by setting $V(x, t) = w(y, \tau)$, $y = \log x, \sigma^2 t = -2\tau$, the equation is turned into

$$-w_\tau + w_{yy} + \Big(\frac{2r}{\sigma^2} - 1\Big)w_y - \frac{2r}{\sigma^2}w = 0 \ .$$

(b) Show that further by setting $w(y,\tau) = e^{\alpha y+\beta\tau}u(y,\tau)$, with suitable α and β, the equation becomes the heat equation

$$u_\tau - u_{yy} = 0 \ .$$

Solution. (a) By a direct computation based on

$$V_x = w_y \frac{1}{x} \ , \quad V_{xx} = w_{yy}\frac{1}{x^2} - w_y\frac{1}{x^2} \ , \quad V_t = \frac{-\sigma^2}{2}w_\tau \ .$$

(b) With a further change of variables, the equation in (a) is transformed into

$$(-\beta u - u_\tau) + (u_{yy} + 2\alpha u_y + \alpha^2 u) + \Big(\frac{2r}{\sigma^2} - 1\Big)(\alpha u + u_y) - \frac{2r}{\sigma^2}u = 0 \ .$$

By choosing α and β according to

$$2\alpha + \frac{2r}{\sigma^2} - 1 = 0 \ , \quad -\beta + \alpha^2 + \alpha\Big(\frac{2r}{\sigma^2} - 1\Big) - \frac{2r}{\sigma^2} = 0 \ ,$$

we obtain the heat equation for u.

Note. The Black-Scholes equation is a model on option pricing. Here V stands for the price of an European put or call. Myron Scholes was awarded the Nobel prize in economics in 1997 together with Robert Merton for proposing this model. Black did not share the honor for he died already.

19. A polynomial P is called a homogeneous polynomial if all terms have the same combined power, that is, there is some m such that $P(tx) = t^m P(x)$ for all $t > 0$. Establish Euler's Identity

$$\sum_{j=1}^n x_j \frac{\partial P}{\partial x_j} = P(x) \ .$$

Verify it for the following homogeneous polynomials:

(a) $x^2 - 3xy + y^2$, and

(b) $x^{15} - x^{10}y^3z^2 + 6y^{14}z$.

Solution. By the homogenity, $P(tx) = t^m P(x)$ for all $t > 0$. Differentiate both sides with respect to t. The left hand side is equal to

$$\frac{\partial}{\partial t}P(tx_1, ..., tx_n) = x_1\frac{\partial P}{\partial x_1}(tx) + ... + x_n\frac{\partial P}{\partial x_n}(tx) \ .$$

The right hand side is $= mt^{m-1}P(x)$. Setting $t = 1$, we have

$$\sum_{j=1}^n x_j \frac{\partial P}{\partial x_j} = P(x) \ .$$

(a) It is equal to $x(2x - 3y) + y(-3x + 2y) = 2(x^2 - 3xy + y^2)$, and

(b) It is equal to $x(15x^{14} - 10x^9y^3z^2) + y(-3x^{10}y^2z^2 + 84y^{13}z) + z(-2x^{10}y^3z + 6y^{14} = 15(x^{15} - x^{10}y^3z^2 + 6y^{14}z)$.

20. An open set D is called connected if for every $x, y \in D$, there exists a parametric curve lying in D connecting x and y. Show that a differentiable function f in an open, connected set with vanishing partial derivatives must be a constant. Hint: Use a regular parametric curve to connect x to y and consider the composite of this curve with f. Chain Rule will do the rest.

Solution. It suffices to show that for any $x, y \in D$, $f(x) = f(y)$. By connectedness of D, there exists a regular curve $\gamma(t)$ such that $\gamma(0) = x$ and $\gamma(1) = y$. Let $\phi(t) = f(\gamma(t)) = f(\gamma_1(t), ..., \gamma_n(t))$. By the chain rule,

$$\phi'(t) = \frac{\partial f}{\partial x_1}(\gamma(t))\gamma_1'(t) + ... + \frac{\partial f}{\partial x_n}(\gamma(t))\gamma_n'(t) = 0 ,$$

for all t. Therefore, $\phi(t)$ is constant function, i.e. $f(\gamma(0)) = f(\gamma(1))$, and hence $f(x) = f(y)$. This implies f is a constant function.

21. Find the directional derivative of each of the following functions at the given point and direction:

(a) $x^2 + y^3 + z^4, \quad (3, 2, 1); \ (-1, 0, 4)/\sqrt{17}$.
(b) $e^{xy} + \sin(x^2 + y^2), \quad (1, -3); \ (1, 1)/\sqrt{2}$.

Solution.

(a)

$$\begin{aligned} D_\xi f &= \xi \cdot \nabla f \\ &= \frac{(-1,0,4)}{\sqrt{17}} \cdot (2x, 3y^2, 4z^3)\Big|_{(3,2,1)} \\ &= \frac{(-1,0,4)}{\sqrt{17}} \cdot (6, 12, 4) \\ &= \frac{10}{\sqrt{17}}. \end{aligned}$$

(b)

$$\begin{aligned} D_\xi f &= \xi \cdot \nabla f \\ &= \frac{(1,1)}{\sqrt{2}} \cdot (ye^{xy} + 2x\cos(x^2+y^2), xe^{xy} + 2y\cos(x^2+y^2))\Big|_{(1,-3)} \\ &= \frac{(1,1)}{\sqrt{2}} \cdot (-3e^{-3} + 2\cos 10, e^{-3} - 6\cos 10) \\ &= \frac{-2e^{-3} - 4\cos 10}{\sqrt{2}} \\ &= -\sqrt{2}(e^{-3} + 2\cos 10). \end{aligned}$$

22. Find the directional derivative of the function $x^2 - y^2$ at $(1, 1)$ whose direction makes an angle of degree 60° with the x-axis.

Solution. The direction is given by $(\cos 60^\circ, \sin 60^\circ) = \frac{1}{2}(1, \sqrt{3})$. Therefore, the directional derivative is $\frac{1}{2}(1, \sqrt{3}) \cdot (2x, -2y)|_{(1,1)} = \frac{1}{2}(1, \sqrt{3}) \cdot (2, -2) = 1 - \sqrt{3}$.

23. Let $g(x, y) = x^2 - xy + y^2$. Find
    (a) the direction along which it increases most rapidly.
    (b) the direction along which it decreases most rapidly.
    (c) the directional at which its directional derivative vanishes.

Solution. We have $\nabla g = (2x - y, -x + 2y)$. Let $|\nabla g| = \sqrt{(2x - y)^2 + (-x + y)^2}$.

(a) The direction is given by $\nabla g/|\nabla g|$.

(b) The direction is given by $-\nabla g/|\nabla g|$.

(c) Let $\xi = (\xi_1, \xi_2)$ be a direction such that $(\xi_1, \xi_2) \cdot (2x - y, -x + 2y) = 0$. We may choose $\xi = (\xi_1, \xi_2) = (-x + 2y, -2x + y)/|\nabla g|$ to have the above equality holds true.

24. Can you find a function whose directional derivative along every direction exists and all equal at $(0, 0)$ but it is not differentiable there ? Hint: An example can be found in a previous problem.

Solution. Consider the function f in Problem 8. Since f eventually vanishes along any direction from $(0, 0)$, $D_\xi f = 0$ for all direction ξ. However, f is not differentiable at $(0, 0)$.

25. (a) Let $f(x, y)$ be a function defined in the first quadrant $\{(x, y) : \ x, y \geq 0\}$. Propose a definition of the partial derivatives of f at $(x, 0), x > 0$ and at $(0, 0)$.
    (b) Let $g(x, y)$ be a function defined in the set $\{(x, y) : \ 0 \leq x \leq y\}$. Propose a definition of the partial derivatives of g at $(0, 0)$.

Solution. (a) For $(x, 0), x > 0$, the partial derivative of f in x is defined as usual, but now the partial derivative in y is

$$\frac{\partial f}{\partial y}(x, 0) = \lim_{h \to 0^+} \frac{f(x, h) - f(x, 0)}{h} \ ,$$

that is, we restrict to the range where y is positive. Similarly, define

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{h \to 0^+} \frac{f(h, 0) - f(0, 0)}{h} \ ,$$

$$\frac{\partial f}{\partial y}(0, 0) = \lim_{h \to 0^+} \frac{f(0, h) - f(x, 0)}{h} \ .$$

(b) Let $\xi = (1, 1)/\sqrt{2}$. Using the relation

$$\frac{\partial f}{\partial \xi} = \frac{\sqrt{2}}{2} \frac{\partial f}{\partial x} + \frac{\sqrt{2}}{2} \frac{\partial f}{\partial y},$$

we define

$$\frac{\partial f}{\partial x}(0, 0) = \sqrt{2} \frac{\partial f}{\partial \xi}(0, 0) - \frac{\partial f}{\partial y}(0, 0) \ .$$

26. Use the differential of an appropriate function to obtain an approximate error estimate and then compare it with the actual one. You may use a calculator.

(a) $\sin 29^\circ \times \tan 46^\circ$.

(b) $\dfrac{1.03^2}{(0.98)^{1/3}(1.05)^{3/4}}$.

(c) $\sqrt{(3.1)^2 + (4.2)^2 + (11.7)^2}$.

Solution.

(a) Let $f(\theta, \phi) = \sin\theta \tan\phi$; $(\theta, \phi) = (30^\circ, 45^\circ) = (\pi/6, \pi/4)$; $(d\theta, d\phi) = (-1^\circ, 1^\circ) = (-\pi/180, \pi/180)$.
Therefore,

$$\begin{aligned} df &= \cos\theta \tan\phi d\theta + \sin\theta \sec^2\phi d\phi \\ &= \frac{\sqrt{3}}{2} \times 1 \times (-\frac{\pi}{180}) + \frac{1}{2} \times \frac{1}{(\frac{\sqrt{2}}{2})^2} \times \frac{\pi}{180} \\ &= (1 - \sqrt{3}/2)\frac{\pi}{180} \ . \end{aligned}$$

and hence the approximate value is given by

$$f(\pi/6, \pi/4) + (1 - \sqrt{3}/2)\frac{\pi}{180} = \frac{1}{2} + (1 - \sqrt{3}/2)\frac{\pi}{180} \ .$$

27. The height and the radius of the base of a cylinder are measured with error up to 0.1 and 0.2 respectively. Find the approximate and exact maximum error of its volume.

Solution. Let the volume be V, the radius be r and the height be h. They are related by

$$V = \pi r^2 h$$

Differentiating both sides yield

$$\begin{aligned} dV &= 2\pi rh dr + \pi r^2 dh \\ &= 2\pi rh(0.1r) + \pi r^2(0.2h) \\ &= 0.4V \end{aligned}$$

Therefore, the approximate error is given by $0.7V/V = 0.4$. The exact error is

$$\pi(r + 0.1r)^2(h + 0.2h) - \pi r^2 h = 0.452V \ .$$

28. A horizontal beam is supported at both ends and supports a uniform load. The deflection at its midpoint is given by

$$S = \frac{k}{wh^3} \ ,$$

where w and h are the width and height respectively of the beam and k is some constant depending on the beam. Show that

$$dS = -S\Big(\frac{1}{w}dw + \frac{3}{h}dh\Big) \ .$$

If $S = 1$ in. when $w = 2$ in. and $h = 4$ in., approximate the deflection when $w = 2.1$ in. and $h = 4.1$ in.. Then compare your approximation with the actual value.

Solution. dS comes from a direct differentiation.

Now since $S(2, 4) = 1$, we find $k = 2 \times 4^3$. Using $dw = 0.1$, $dh = 0.1$,

$$dS = -S\Big(\frac{0.1}{2} + \frac{3}{4}0.1\Big) = -0.125 \ .$$

The exact error is

$$\frac{k}{2.1 \times (4.1)^3} - 1 = -0.1156 \ .$$

29. The point $(1, 2)$ lies on the curve defined by the equation

$$f(x, y) = 2x^3 + y^3 - 5xy = 0 \ .$$

Approximate the y-coordinate of the nearby point (x, y) on this curve which $x = 1.2$.

Solution. The relation defined a function $y = g(x)$:

$$2x^3 + g^3(x) - 5xg(x) = 0 \ .$$

The y-coordinate is given by $g(1.2)$. Now we will obtain the approximate value by using the differential at $x = 1$. Differentiating it in x:

$$6x^2 + 3g^2(x)g'(x) - 5g(x) - 5xg'(x) = 0,$$

which gives

$$g'(x) = \frac{6x^2 - 5g(x)}{5x - 3g^2(x)} \ .$$

So $g'(1) = 4/7$ and $dg = g'(1)(1.2 - 1) = 4/35 = 0.1143$. Therefore, the approximate y-coordinate is given by 2.1143.

30. Suppose that $T = x(e^y + e^{-y})$ where $x = 2, y = \log 2$ with maximum possible errors 0.1 in x and 0.02 in y. Estimate the maximum possible induced error in the computed value of T.

Solution. We have

$$\begin{aligned} dT &= (e^y + e^{-y})dx + x(e^y - e^{-y})dy \\ &= \left(2 + \frac{1}{2}\right) \times 0.1 + 2\left(2 - \frac{1}{2}\right) \times 0.02 \\ &= 0.31 \ . \end{aligned}$$